B.Tech I Year

Regular Course Handbook

Subject Name: Engineering Mathematics-II (Unit-II)

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BAS203 : ENGINEERING MATHEMATICS-II

Course Objectives:

The objective of this course is to familiarize the prospective engineers with techniques in ordinary differential equations, Laplace transform, sequence and series,Fourier series and complex variables. It aims to equip the students to deal with advanced level of mathematics and applications that would be essential for their disciplines.

The students will learn:

- The effective mathematical tools for the solutions of differential equations that model physical processes.
- The basic knowledge of Laplace transform and its applications in solving differential equations.
- The tool for convergence of series and expansion of function using Fourier series for learning advanced Engineering Mathematics.
- The tools of differentiation of functions of complex variables that are used in various techniques dealing with engineering problems.
- The tools of integration of functions of complex variables that are used in various techniques dealing with engineering problems.

Content	Contact Hours
Unit -1: Ordinary Differential Equation of Higher Order	8
Linear differential equation of nth order with constant coefficients, Simultaneous linear differential equations, Second order linear differential equations with variable coefficients, Solution by changing independent variable, Method of variation of parameters, Cauchy-Euler equation, Application of differential equations in solving engineering problems.	
Unit-2: Laplace Transform	10
Laplace transform, Existence theorem, Properties of Laplace Transform, Laplace transform of derivates and integrals, Unit step function, Laplace transform of periodic function, Inverse Laplace transform, Convolution theorem. Application of Laplace Transform to solve ordinary differential equations and simultaneous differential equations.	
Unit-3: Sequence and Series	8
Definition of Sequence and series with examples, Convergence of series, Tests for convergence of series, Ratio test, D' Alembert's test, Raabe's test, Comparison test. Fourier series, Half range Fouriersine and cosine series.	
Unit-4: Complex Variable–Differentiation	8
Functions of complex variable, Limit, Continuity and differentiability, Analytic functions, Cauchy- Riemann equations (Cartesian and Polar form), Harmonic function, Method to find Analytic functions, Milne's Thompson Method, Conformal mapping, Mobius transformation and their properties.	
Unit-5: Complex Variable –Integration	8
Complex integration, Cauchy- Integral theorem, Cauchy integral formula, Taylor's and Laurent's series, singularities and itsclassification, zeros of analytic functions, Residues, Cauchy's Residue theorem and its application.	

Course Outcomes:

	Course Outcome (CO)	Bloom's Level
At the end of this course, the students will be able to:		
CO 1	Remember the concept differentiation to evaluate LDE of nth order with constant coefficient and LDE with variable coefficient of 2nd order.	K1 &K5
CO 2	Understand and apply the concept of Laplace Transform to evaluate differential equations	K2 ,K3 & K5
CO 3	Understand the concept of convergence to analyze the convergence of series and expansion of the function for Fourier series.	K2 & K4
CO 4	Apply the concept of analyticity, Harmonic function and create theimage of function applying conformal transformation	K3, K6& K3
CO 5	Apply the concept of Cauchy Integral theorem, Cauchy Integralformula, singularity and calculus of residue to evaluate integrals	K3& K5

K1 – Remember, K2 – Understand, K3 – Apply, K4 – Analyze, K5 – Evaluate, K6 – Create

Text Books:

1. B. V. Ramana, Higher Engineering Mathematics, Tata McGraw-Hill Publishing Company Ltd., 2008.
2. B. S. Grewal, Higher Engineering Mathematics, Khanna Publisher, 2005.
3. R. K. Jain & S. R. K. Iyenger, Advance Engineering Mathematics , Narosa Publishing - House, 2002

Reference Books:

1. E. Kreyszig, Advance Engineering Mathematics, John Wiley & Sons, 2005.
2. Peter V. O'Neil, Advance Engineering Mathematics, Thomson (Cengage) Learning, 2007.
3. Maurice D. Weir, Joel Hass, Frank R.Giordano, Thomas, Calculus, Eleventh Edition, Pearson.
4. G.B Thomas, R L Finney, Calculus and Analytical Geometry, Ninth Edition Pearson, 2002.
5. James Ward Brown and Ruel V Churchill, Fourier Series and Boundary Value Problems, 8th Edition-McGraw-Hill
6. D. Poole, Linear Algebra: A Modern Introduction, 2nd Edition, Brooks/Cole, 2005.
7. Veerarajan T., Engineering Mathematics for first year, McGraw-Hill, New Delhi, 2008.
8. Charles E Roberts Jr, Ordinary Diffrential Equations, Application, Model and Computing, CRC Press T&F Group.
9. Ray Wylie C and Louis C Barret, Advanced Engineering Mathematics, 6thEdition, McGraw-Hill.
10. James Ward Brown and Ruel V Churchill, Complex Variable and Applications, 8th Edition, McGraw-Hill.
11. P. Sivaramakrishna Das and C. Vijayakumari, Engineering Mathematics, 1st Edition, Pearson India Education Services Pvt. Ltd.
12. Advanced Engineering Mathematics By Chandrika Prasad, Reena Garg Khanna Publishing House, Delhi.
13. Laplace Transforms by Schaum's series, 2005 Edition, Spiegel Publicaiton.

Laplace Transform   Unit : 02

Defn (Laplace Transform): Let $F(t)$ be a function of t defined for all $t \geq 0$. Then the Laplace transform of $F(t)$, denoted by $L\{F(t)\}$, is defined by

$$\boxed{L\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\,dt}$$

provided that the integral exists, 'p' is a parameter which may be real or complex.

The function $f(p)$ is called the Laplace transform.

NOTE: $L\{F(t)\}$ is said to exists if the above integral converges for some value of P otherwise not.

Linearity properties :

If c_1, c_2 are constants and f, g are function of t, then

$$\boxed{L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\}}$$

Functions of Exponential Order: A function $F(t)$ is said to be of exponential order as $t \to \infty$, if there exist constants M and b and a fixed value t_0 of t such that

$$|F(t)| < Me^{bt}, \text{ for } t \geq t_0$$

Existence Theorem:

If $F(t)$ is piecewise continuous for $t \geq 0$ and is of exponential order b, then

$$L\{F(t)\} = f(p) \text{ exists for } p > b.$$

Laplace Transform of some elementary functions

① $L\{1\} = \frac{1}{p}$, $p>0$

Pf $\Rightarrow$ $L\{1\} = \int_0^\infty e^{-pt}(1)dt = \left[\frac{e^{-pt}}{-p}\right]_0^\infty = \left(\frac{-1}{p}\right)(0-1)$

$$\boxed{L\{1\} = \frac{1}{p}}, \quad p>0$$

② $L\{t^n\} = \frac{n!}{p^{n+1}}$ or $\frac{\Gamma(n+1)}{p^{n+1}}$

Pf $L\{t^n\} = \int_0^\infty e^{-pt}t^n dt = \int_0^\infty e^{-x}\left(\frac{x}{p}\right)^n \frac{dx}{p}$

Let $pt = x$
$\Rightarrow p\,dt = dx$
$\Rightarrow dt = \frac{dx}{p}$

$= \frac{1}{p^{n+1}}\int_0^\infty e^{-x}x^n dx$

$= \frac{1}{p^{n+1}}\int_0^\infty e^{-x}x^{(n+1)-1}dx = \frac{\Gamma(n+1)}{p^{n+1}}$

$\boxed{L\{t^n\} = \Gamma(n+1)/p^{n+1}}$ $\boxed{L\{t^n\} = \frac{n!}{p^{n+1}}, n \in \mathbb{N}}$

③ $L\{e^{at}\} = \frac{1}{p-a}$, $p>a$

Pf $L\{e^{at}\} = \int_0^\infty e^{-pt}e^{at}dt = \int_0^\infty e^{-(p-a)t}dt$

$= \left[\frac{e^{-(p-a)t}}{-(p-a)}\right]_0^\infty = \frac{-1}{p-a}(0-1) = \frac{1}{p-a}$, $p>a$

④ $L\{\sin at\} = \frac{a}{p^2+a^2}$, $p>0$

⑤ $L\{\cos at\} = \frac{p}{p^2+a^2}$, $p>0$

⑥ $L\{\sinh at\} = \frac{a}{p^2-a^2}$, $p>|a|$

⑦ $L\{\cosh at\} = \frac{p}{p^2-a^2}$, $p>|a|$

$F(t)$	$L\{F(t)\} = f(p)$
1	$\frac{1}{p}$, $p>0$
t	$\frac{1}{p^2}$, $p>0$
t^n, $n \in \mathbb{N}$	$\frac{n!}{p^{n+1}}$, $p>0$
t^n, $n>-1$	$\Gamma(n+1)/p^{n+1}$, $p>0$
e^{at}	$1/(p-a)$, $p>a$
e^{-at}	$1/(p+a)$
$\sin at$	$a/(p^2+a^2)$, $p>0$
$\cos at$	$p/(p^2+a^2)$, $p>0$
$\sinh at$	$a/(p^2-a^2)$, $p>\|a\|$
$\cosh at$	$p/(p^2-a^2)$, $p>\|a\|$

Ques Find the Laplace transform of

$$7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t+2$$

Ans $L(7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t+2)$

$$\Rightarrow 7L\{e^{2t}\}+9L\{e^{-2t}\}+5L\{\cos t\}+7L\{t^3\}+5L\{\sin 3t\}+2L\{1\}$$

$$\Rightarrow \frac{7}{p-2}+\frac{9}{p+2}+\frac{5p}{p^2+1}+7\cdot\frac{3!}{t^4}+5\cdot\frac{3}{p^2+9}+2\cdot\frac{1}{p}$$

$$\Rightarrow \frac{7}{p-2}+\frac{9}{p+2}+\frac{5p}{p^2+1}+\frac{42}{t^4}+\frac{15}{p^2+9}+\frac{2}{p}$$

Ques Find the Laplace transform of

$$F(t)=\begin{cases}\cos t, & 0<t<\pi\\ 0, & t>\pi\end{cases}$$

Ans $L\{F(t)\}=\int_0^\infty e^{-pt}F(t)\,dt$

$$=\int_0^\pi e^{-pt}F(t)\,dt+\int_\pi^\infty e^{-pt}F(t)\,dt$$

$$=\int_0^\pi e^{-pt}\cos t\,dt+\int_\pi^\infty e^{-pt}(0)\,dt$$

$$=\left[\frac{e^{-pt}}{(-p)^2+1}\left[-p\cos t+\sin t\right]\right]_0^\pi+0$$

$$\boxed{L\{F(t)\}=\left[\frac{e^{-pt}}{p^2+1}\left[-p\cos t+\sin t\right]\right]_0^\pi}$$

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}\left[a\cos bx+b\sin bx\right]$$

$$L\{F(t)\}=\frac{e^{-p\pi}}{p^2+1}[p]-\frac{1}{p^2+1}[-p]=\frac{p(1+e^{-p\pi})}{p^2+1}.$$

First Shifting property or first translation property

If $L\{F(t)\}=f(p)$ then $\boxed{L\{e^{at}F(t)\}=f(p-a)}$

Proof $L\{e^{at}F(t)\}=\int_0^\infty e^{-pt}e^{at}F(t)\,dt$

$$=\int_0^\infty e^{-(p-a)t}F(t)\,dt=f(p-a)$$

Ques Find the Laplace transform of

$$e^{-3t}(\cos 4t+3\sin 4t)$$

Ans $f(t)=\cos 4t+3\sin 4t$

then $L\{F(t)\}=L\{\cos 4t+3\sin 4t\}$

$$=L\{\cos 4t\}+3L\{\sin 4t\}$$

$$\boxed{L\{F(t)\}=\frac{p}{p^2+16}+\frac{12}{p^2+16}}=f(p)$$

Now by first shifting property

$$L\{e^{-3t}F(t)\}=f(p+3)$$

$$=\frac{p+3}{(p+3)^2+16}+\frac{12}{(p+3)^2+16}$$

$$=\frac{p+15}{p^2+6p+25}$$

Ques. find the Laplace transform $\cosh at \cos bt$ (2016)

Ans We have to find Laplace transform of

$$\cosh at \cos bt = \left(\frac{e^{at}+e^{-at}}{2}\right)\cos bt$$

$$= \frac{1}{2}\left[e^{at}\cos bt + e^{-at}\cos bt\right]$$

$$= I_1 + I_2 \quad \text{——— (1)}$$

Now

$$L\{\cos bt\} = \frac{p}{p^2+b^2} = f(p)$$

then $L\{e^{at}\cos bt\} = f(p-a) = \dfrac{p-a}{(p-a)^2+b^2}$

$L\{e^{-at}\cos bt\} = f(p+a) = \dfrac{p+a}{(p+a)^2+b^2}$ [first Shifting property]

Now using (1)

$$L\{\cosh at \cos bt\} = \frac{1}{2}\left[L\{e^{at}\cos bt\} + L\{e^{-at}\cos bt\}\right]$$

$$= \frac{1}{2}\left[\frac{p-a}{(p-a)^2+b^2} + \frac{p+a}{(p+a)^2+b^2}\right]$$

Second Translation Property or Heaviside's Shifting Theorem.

If $L\{F(t)\} = f(p)$ and $G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$

then $\boxed{L\{G(t)\} = e^{-ap} f(p)}$

Pf. $L\{G(t)\} = \int_0^\infty e^{-pt} G(t)\,dt$

$$= \int_0^a e^{-pt} G(t)\,dt + \int_a^\infty e^{-pt} G(t)\,dt$$

$$= \int_a^\infty e^{-pt} F(t-a)\,dt + \int_0^a e^{-pt}\, 0\,dt$$

$$= \int_a^\infty e^{-pt} F(t-a)\,dt$$

Let $t-a=u \Rightarrow dt = du$

$$= \int_0^\infty e^{-p(u+a)} F(u)\,du$$

$$= e^{-pa}\int_0^\infty e^{-pu} F(u)\,du = e^{-pa}\int_0^\infty e^{-pt}F(t)\,dt$$

$$\boxed{L\{G(t)\} = e^{-ap} f(p)}$$

Que. find the Laplace transform of the function

(1) $F(t) = \begin{cases} e^{t-a}, & t>a \\ 0, & t<a \end{cases}$ (2012)

Ans $L\{e^t\} = \dfrac{1}{p-1}$, $p>1$ so $f(p) = \dfrac{1}{p-1}$

then by using Second Shifting property

$$L\{F(t)\} = e^{-ap} f(p) = \frac{e^{-ap}}{p-1}, \quad p>1$$

(ii) $F(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$

Sol $L\{\cos t\} = \frac{p}{p^2+1} = f(p)$

So $L\{F(t)\} = e^{-(2\pi/3)p} f(p) = \frac{\left(e^{-\left(\frac{2\pi}{3}\right)p}\right) p}{(p^2+1)}$

Change of Scale Property

If $L\{F(t)\} = f(p)$ then $\boxed{L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)}$

Que If $L\{F(t)\} = \frac{p^2 - p + 1}{(2p+1)^2(p-1)}$, Show that (2014, 18)

$L\{F(2t)\} = \frac{p^2 - 2p + 4}{4(p+1)^2(p-2)}$

Sol $L\{F(t)\} = \frac{p^2 - p + 1}{(2p+1)^2(p-1)} = f(p)$

Now $L\{F(2t)\} = \frac{1}{2} f\left(\frac{p}{2}\right)$ [Using Change of Scale property]

$L\{F(2t)\} = \frac{1}{2}\left[\frac{\left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right) + 1}{\left(2\left(\frac{p}{2}\right)+1\right)^2\left(\frac{p}{2} - 1\right)}\right] = \frac{1}{2}\left[\frac{\frac{p^2}{4} - \frac{p}{2} + 1}{(p+1)^2\left(\frac{p-2}{2}\right)}\right]$

$L\{F(2t)\} = \frac{1}{2}\left[\frac{\frac{p^2 - 2p + 4}{4}}{(p+1)^2\left(\frac{p-2}{2}\right)}\right] = \frac{p^2 - 2p + 4}{4(p+1)^2(p-2)}$

Laplace Transform of Derivatives

$L\{F^{(n)}(t)\} = p^n L\{F(t)\} - p^{n-1} F(0) - p^{n-2} F'(0) - \cdots - F^{(n-1)}(0)$

In particular.

for n=1 $\quad L\{F'(t)\} = p\,L\{F(t)\} - F(0)$
$= p f(p) - F(0)$

for n=2 $\quad L\{F''(t)\} = p^2 L\{F(t)\} - p\,F(0) - F'(0)$

for n=3 $\quad L\{F'''(t)\} = p^3 L\{F(t)\} - p^2 F(0) - pF'(0) - F''(0)$

Laplace transform of Integrals

If $L\{F(t)\} = f(p)$, then $L\left\{\int_0^t F(t)\,dt\right\} = \frac{1}{p} f(p)$

Multiplication by t^n

If $L\{F(t)\} = f(p)$ then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p)$

In particular.

$L\{t F(t)\} = -\frac{d}{dp} f(p) = -f'(p)$

$L\{t^2 F(t)\} = \frac{d^2}{dp^2} f(p) = f''(p)$

$L\{t^3 F(t)\} = -\frac{d^3}{dp^3} f(p) = -f'''(p)$

Divison by t

If $L\{F(t)\} = f(p)$ then $\boxed{L\{\frac{1}{t}F(t)\} = \int_p^{\infty} f(p)\,dp}$

Ques find the Laplace transform of

(i) $\int_0^t e^{-t}\cos t\,dt$ (2022)

(ii) $\int_0^t \frac{\sin t}{t}\,dt$ (2022)

(iii) $\int_0^t e^{t}\frac{\sin t}{t}\,dt$ (2014)

Sol (i) $L\{\cos t\} = \frac{p}{p^2+1}$

$L\{e^{-t}\cos t\} = \frac{p+1}{(p+1)^2+1} = f(p)$

Now

$L\left(\int_0^t e^{-t}\cos t\,dt\right) = \frac{1}{p}f(p)$

(by Laplace transform of Integral)

$L\left(\int_0^t e^{-t}\cos t\,dt\right)$

$= \frac{1}{p}\left[\frac{p+1}{(p+1)^2+1}\right]$

$= \frac{p+1}{p(p^2+2p+2)}$

(ii) $L\{\sin t\} = \frac{1}{p^2+1}$

$L\left\{\frac{\sin t}{t}\right\} = \int_p^{\infty}\frac{1}{p^2+1}\,dp$

$= (\tan^{-1}p)_p^{\infty}$

$= (\tan^{-1}\infty - \tan^{-1}p)$

$= \frac{\pi}{2} - \tan^{-1}p$

$L\left\{\frac{\sin t}{t}\right\} = \cot^{-1}p = f(p)$

Now

$L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{p}f(p)$

$= \frac{\cot^{-1}p}{p}$

(iii) from part (ii) $L\left\{\frac{\sin t}{t}\right\} = \cot^{-1}p$

$\Rightarrow L\left\{e^{t}\frac{\sin t}{t}\right\} = \cot^{-1}(p-1) = f(p)$

$\Rightarrow L\left\{\int_0^t e^{t}\frac{\sin t}{t}\,dt\right\} = \frac{1}{p}f(p) = \frac{\cot^{-1}(p-1)}{p}$

Ques find the Laplace transform.

(i) $te^{-t}\sin 2t$ (2010)

(ii) $t^2e^{t}\sin 4t$ (2011)

Sol (i) $L\{te^{-t}\sin 2t\} = ?$

$L\{\sin 2t\} = \frac{2}{p^2+4}$

$L\{e^{-t}\sin 2t\} = \frac{2}{(p+1)^2+4}$

$= \frac{2}{p^2+2p+5} = f(p)$

$L\{te^{-t}\sin 2t\} = -\frac{d}{dp}f(p)$

$= -\frac{d}{dp}\left[\frac{2}{p^2+2p+5}\right]$

$= (-2)\left[\frac{-1}{(p^2+2p+5)^2}(2p+2)\right]$

$= \frac{4p+4}{(p^2+2p+5)^2}$

(ii) $L\{t^2e^{t}\sin 4t\} = ?$

$L\{\sin 4t\} = \frac{4}{p^2+16} = f(p)$

$L\{t^2\sin 4t\} = \frac{d^2}{dp^2}f(p)$

$= \frac{d^2}{dp^2}\left[\frac{4}{p^2+16}\right]$

$= \frac{d}{dp}\left[\frac{-8p}{(p^2+16)^2}\right]$

$= -8\left[\frac{16-3p^2}{(p^2+16)^3}\right]$

Now

$L\{t^2e^{t}\sin 4t\} = -8\left[\frac{16-3(p-1)^2}{((p-1)^2+16)^3}\right]$

$= \frac{8(3p^2-6p-13)}{(p^2-2p+17)^3}$

Q find the following integral.

(i) $\int_0^\infty t^3 e^{-t} \sin t \, dt$ (ii) $\int_0^\infty \frac{e^{-t} \sin^2 t}{t} dt$

Ans (i) $L\{\sin t\} = \frac{1}{p^2+1}$

$L\{t^3 \sin t\} = -\frac{d^3}{dp^3}\left[\frac{1}{p^2+1}\right]$

$= \frac{d^2}{dp^2}\left[\frac{2p}{(p^2+1)^2}\right]$

$= \frac{d}{dp}\left[\frac{2(1-3p^2)}{(p^2+1)^3}\right]$

$= \frac{24p(p^2-1)}{(p^2+1)^4}$

$L\{t^3 \sin t\} = \frac{24p(p^2-1)}{(p^2+1)^4}$

$\int_0^\infty e^{-pt} t^3 \sin t \, dt = \frac{24p(p^2-1)}{(p^2+1)^4}$

by defn of Laplace Trans^n

Now put $p=1$ then

$\int_0^\infty e^{-t} t^3 \sin t \, dt = \frac{24(1-1)}{(1+1)^4}$

$= 0$

(ii) $L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$

$= \frac{1}{2} L\{1-\cos 2t\}$

$= \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$

$= \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right]$

Now

$L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{2}\int_p^\infty \left(\frac{1}{p} - \frac{p}{p^2+4}\right) dp$

$= \frac{1}{2}\left[\log p - \frac{1}{2}\log(p^2+4)\right]_p^\infty$

$= \frac{1}{4}\left\{\log\left[\frac{p^2}{p^2+4}\right]\right\}_p^\infty$

$= \frac{1}{4}\left[\lim_{p\to\infty} \log\left[\frac{p^2}{p^2+4}\right] - \log\left[\frac{p^2}{p^2+4}\right]\right]$

$= \frac{1}{4}\left[\lim_{p\to\infty} \log\left[\frac{1}{1+\frac{4}{p^2}}\right] + \log\left[\frac{p^2+4}{p^2}\right]\right]$

$= \frac{1}{4}\left[0 + \log\left[\frac{p^2+4}{p^2}\right]\right]$

$= \frac{1}{4}\log\left[\frac{p^2+4}{p^2}\right]$

$L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\log\left[\frac{p^2+4}{p^2}\right]$

$\int_0^\infty e^{-pt} \frac{\sin^2 t}{t} dt = \frac{1}{4}\log\left(\frac{p^2+4}{p^2}\right)$

Put $p=1$ $\boxed{\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt = \frac{1}{4}\log 5}$

Que Find the Laplace transform.

(i) $\frac{e^{-at} - e^{-bt}}{t}$ (ii) $\frac{\cos at - \cos bt}{t}$

A (i) $L\{e^{-at} - e^{-bt}\}$

$= \frac{1}{p+a} - \frac{1}{p+b}$

$L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \int_p^\infty \left(\frac{1}{p+a} - \frac{1}{p+b}\right) dp$

$= \left[\log(p+a) - \log(p+b)\right]_p^\infty$

$= \left[\log\left(\frac{p+a}{p+b}\right)\right]_p^\infty$

$= \lim_{p\to\infty} \log\left(\frac{p+a}{p+b}\right) - \log\left(\frac{p+a}{p+b}\right)$

$= \lim_{p\to\infty} \log\left(\frac{1+\frac{a}{p}}{1+\frac{b}{p}}\right) + \log\left(\frac{p+b}{p+a}\right)$

$= 0 + \log\left(\frac{p+b}{p+a}\right)$

$= \log\left(\frac{p+b}{p+a}\right)$

(ii) $L\{\cos at - \cos bt\}$

$= \frac{p}{p^2+a^2} - \frac{p}{p^2+b^2}$

$L\{\cos at - \cos bt\} = \int_p^\infty \left(\frac{p}{p^2+a^2} - \frac{p}{p^2+b^2}\right) dp$

$= \frac{1}{2}\left[\log(p^2+a^2) - \log(p^2+b^2)\right]_p^\infty$

$= \frac{1}{2}\left[\log\left(\frac{p^2+a^2}{p^2+b^2}\right)\right]_p^\infty$

$= \lim_{p\to\infty} \frac{1}{2}\log\left(\frac{p^2+a^2}{p^2+b^2}\right) - \frac{1}{2}\log\left(\frac{p^2+a^2}{p^2+b^2}\right)$

$= \lim_{p\to\infty} \frac{1}{2}\log\left[\frac{1+\frac{a^2}{p^2}}{1+\frac{b^2}{p^2}}\right] + \frac{1}{2}\log\left(\frac{p^2+b^2}{p^2+a^2}\right)$

$= 0 + \frac{1}{2}\log\left[\frac{p^2+b^2}{p^2+a^2}\right]$

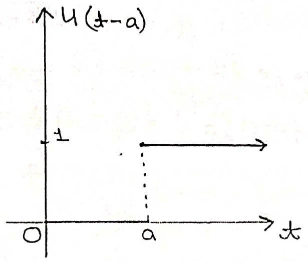
$= \frac{1}{2}\log\left[\frac{p^2+b^2}{p^2+a^2}\right]$

Unit Step function (or Heaviside's Unit step function)

The unit step function $u(t-a)$ is defined as

$$u(t-a) = \begin{cases} 0, & \text{for } t<a \\ 1, & \text{for } t \geq a \end{cases}$$

where $a \geq 0$



As a particular case,

$$u(t) = \begin{cases} 0, & \text{for } t<0 \\ 1, & \text{for } t \geq 0 \end{cases}$$

The product $F(t)\, u(t-a) = \begin{cases} 0, & \text{for } t<a \\ F(t), & \text{for } t \geq a \end{cases}$

The function $F(t-a)\, u(t-a)$ represents the graph of $F(t)$ shifted through a distance a to the right.

Laplace transform of Unit step function

$$L\{u(t-a)\} = \int_0^\infty e^{-pt} u(t-a)\, dt$$

$$= \int_0^a e^{-pt}(0)\, dt + \int_a^\infty e^{-pt}(1)\, dt$$

$$= 0 + \left[\frac{e^{-pt}}{-p}\right]_a^\infty = \frac{1}{p} e^{-ap}$$

so $\boxed{L\{u(t-a)\} = \frac{1}{p} e^{-ap}}$

In particular $\boxed{L\{u(t)\} = \frac{1}{p}}$

Second Shifting Theorem

If $L\{F(t)\} = f(p)$, then $\boxed{L\{F(t-a).u(t-a)\} = e^{-ap} f(p)}$

Ques Find the Laplace transform of

(i) $(t-1)^2 u(t-1)$

On comparing given function with $F(t-a)\, u(t-a)$, we get

$a=1$, $F(t) = t^2$

so $L\{F(t)\} = L\{t^2\} = \frac{2}{p^3}$

$L\{F(t)\} = \frac{2}{p^3} = f(p)$

$L\{F(t-a) u(t-a)\} = e^{-ap} f(p)$

so

$L\{(t-1)^2 u(t-1)\} = e^{-p}\left(\frac{2}{p^3}\right)$

$= \frac{2e^{-p}}{p^3}$

(ii) $\sin(t-2)\, u(t-2)$

On comparing given fun with $F(t-a)\, u(t-a)$, we get

$a=2$, $F(t) = \sin t$

So $L\{F(t)\} = L\{\sin t\}$

$= \frac{1}{p^2+1} = f(p)$

$L\{F(t-a) u(t-a)\} = e^{-ap} f(p)$

so

$L\{\sin(t-2) u(t-2)\} = e^{-2p}\left[\frac{1}{p^2+1}\right]$

$= \frac{e^{-2p}}{p^2+1}$

(iii) $\sin t \; u(t-\pi)$

$\sin t = \sin[(t-\pi)+\pi]$

$\sin t = -\sin(t-\pi)$

$L\{\sin t \; u(t-\pi)\}$

$= L\{-\sin(t-\pi)\, u(t-\pi)\}$

$= -L\{\sin(t-\pi)\, u(t-\pi)\}$

On comparing $\sin(t-\pi)u(t-\pi)$ with $F(t-a)u(t-a)$ we get

$a=\pi$, $F(t)=\sin t$

$L\{\sin t \, u(t-\pi)\} = -e^{-\pi p} L\{\sin t\}$

$= \dfrac{-e^{-\pi p}}{p^2+1}$

(iv) $e^{-3t}u(t-2)$

$L\{u(t-2)\} = \frac{1}{p}e^{-2p} = f(p)$

$\left(\because L\{u(t-a)\} = \frac{1}{p}e^{-ap}\right)$

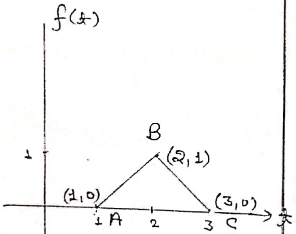
$L\{e^{-3t}u(t-2)\}$

$= f(p+3)$

$= \dfrac{1}{(p+3)} e^{-2(p+3)}$

$\left(\because L\{e^{-at}F(t)\} = f(p+a)\right)$

Ex^m Express the function shown in the diagram in terms of unit step function and obtain its Laplace transform (2015)



Ans Equation of line AB

$f(t) = t-1, \quad 1<t<2$

Equation of line BC

$f(t) = 3-t, \quad 2<t<3$

So $f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$

$\therefore F(t) = (t-1)\{u(t-1) - u(t-2)\} + (3-t)\{u(t-2) - u(t-3)\}$

$= (t-3)u(t-3) - 2(t-2)u(t-2) + (t-1)u(t-1)$

Hence

$L\{F(t)\} = L\{(t-3)u(t-3)\} - 2L\{(t-2)u(t-2)\} + L\{(t-1)u(t-1)\}$

$= \dfrac{e^{-3p}}{p^2} - \dfrac{2e^{-2p}}{p^2} + \dfrac{e^{-p}}{p^2}$ [By Second Shifting property]

$= \dfrac{e^{-p}(e^{-2p} - 2e^{-p} + 1)}{p^2} = \dfrac{e^{-p}\{(e^{-p})^2 - 2e^{-p} + 1^2\}}{p^2}$

$$\boxed{L\{F(t)\} = \dfrac{e^{-p}(1-e^{-p})^2}{p^2}}$$

Que Express the following function in terms of unit step function and find its Laplace transform.

Equation of line OA

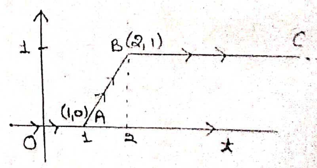
$f(t) = 0, \quad 0<t<1$

Equation of line AB

$f(t) = t-1, \quad 1<t<2$

Equation of line BC

$f(t) = 1, \quad t>2$



So $f(t) = \begin{cases} 0, & 0<t<1 \\ t-1, & 1<t<2 \\ 1, & t>2 \end{cases}$

So $f(t) = (t-1)[u(t-1)-u(t-2)] + 1[u(t-2)]$

$f(t) = (t-1)u(t-1) - (t-2)u(t-2)$.

By Second Shifting theorem

$$L\{f(t)\} = L\{(t-1)u(t-1)\} - L\{(t-2)u(t-2)\}$$

$$= \frac{e^{-p}}{p^2} - \frac{e^{-2p}}{p^2} = \left(\frac{e^{-p}-e^{-2p}}{p^2}\right)$$

Periodic function [Laplace transform of periodic function]

If $f(t)$ is a periodic function with period T

i.e. $f(t+T) = f(t)$ then

$$\boxed{L\{f(t)\} = \frac{1}{1-e^{-pT}}\int_0^T e^{-pt} f(t)\,dt}$$

Ques Draw the graph and find the Laplace transform of the triangular wave function of period $2c$ given by

$$f(t) = \begin{cases} t, & 0 < t \le c \\ 2c-t, & c < t < 2c \end{cases} \qquad (2014, 22)$$

Ans Here period $\boxed{T = 2c}$

$$L\{f(t)\} = \frac{1}{1-e^{-pT}}\int_0^T e^{-pt} f(t)\,dt$$

$$= \frac{1}{1-e^{-2cp}}\int_0^{2c} e^{-pt} f(t)\,dt$$

$$= \frac{1}{1-e^{-2cp}}\left[\int_0^c e^{-pt}\,t\,dt + \int_c^{2c} e^{-pt}(2c-t)\,dt\right]$$

$$= \frac{1}{1-e^{-2cp}}\left[\left\{\frac{t e^{-pt}}{(-p)} - \frac{e^{-pt}}{(-p)^2}\right\}_0^c + \left\{(2c-t)\frac{e^{-pt}}{(-p)} - (-1)\frac{e^{-pt}}{(-p)^2}\right\}_c^{2c}\right]$$

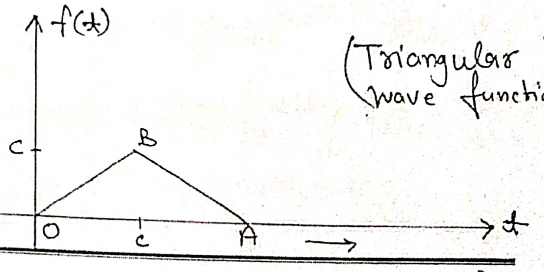
$$= \frac{1}{1-e^{-2cp}}\left[\left\{-\frac{ce^{-cp}}{p} - \frac{e^{-cp}}{p^2} + \frac{1}{p^2}\right\} + \left\{\frac{e^{-2cp}}{p^2} + \frac{ce^{-cp}}{p} - \frac{e^{-cp}}{p^2}\right\}\right]$$

$$= \frac{1}{1-e^{-2cp}}\left[\frac{1-2e^{-cp}+e^{-2cp}}{p^2}\right]$$

$$= \frac{1}{1-(e^{-cp})^2}\left[\frac{(1-e^{-cp})^2}{p^2}\right]$$

$$= \frac{1}{p^2}\left(\frac{1-e^{-cp}}{1+e^{-cp}}\right) = \frac{1}{p^2}\left[\frac{e^{\frac{cp}{2}} - e^{-\frac{cp}{2}}}{e^{\frac{cp}{2}} + e^{-\frac{cp}{2}}}\right]$$

$$\boxed{L\{f(t)\} = \frac{1}{p^2}\tanh\left(\frac{cp}{2}\right)}$$



(Triangular wave function)

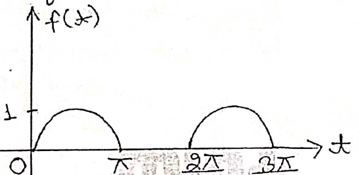
Ques Find the Laplace transform of the following rectified semi-wave function defined by

$$f(t) = \begin{cases} \sin(\omega t), & 0 < t \le \pi/\omega \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases} \quad (2010/2011)$$

or

Find the Laplace transform of following periodic function.

(Half-wave rectifier)



Ans Here $f(t)$ is a periodic function with period $\frac{2\pi}{\omega}$.

so $\boxed{T = 2\pi/\omega}$

$$\therefore L\{F(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi p/\omega}} \int_0^{2\pi/\omega} e^{-pt} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi p/\omega}} \left[\int_0^{\pi/\omega} e^{-pt} \sin \omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-pt}(0)\,dt \right]$$

$$= \frac{1}{1-e^{-2\pi p/\omega}} \int_0^{\pi/\omega} e^{-pt} \sin \omega t\,dt + 0$$

$$= \frac{1}{1-e^{-2\pi p/\omega}} \left[\frac{e^{-pt}(-p\sin \omega t - \omega \cos \omega t)}{p^2+\omega^2} \right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi p/\omega}} \left[\frac{\omega e^{-\pi p/\omega} + \omega}{p^2+\omega^2} \right]$$

$$= \frac{\omega(1+e^{-\pi p/\omega})}{(1+e^{-\pi p/\omega})(1-e^{-\pi p/\omega})(p^2+\omega^2)}$$

$$= \frac{\omega}{(1-e^{-\pi p/\omega})(p^2+\omega^2)}$$

Ques Find the Laplace transform of a periodic function "saw-tooth wave" function with period 1 and defined as $f(t) = kt$ in $0 < t < 1$ (2017)

Ans Given function is saw-tooth function with period $T = 1$ so

$$L\{f(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\,dt.$$

$$= \frac{1}{1-e^{-p}} \int_0^1 e^{-pt}(kt)\,dt$$

$$= \left(\frac{k}{1-e^{-p}}\right) \int_0^1 e^{-pt}\, t\,dt$$

$$= \frac{k}{1-e^{-p}} \left[-\frac{t}{p} e^{-pt} - \frac{1}{p^2} e^{-pt} \right]_0^1$$

$$= \frac{k}{1-e^{-p}} \left[-\frac{1}{p} e^{-p} - \frac{1}{p^2} e^{-p} + 0 + \frac{1}{p^2} \right]$$

$$= \frac{k}{(1-e^{-p})} \left[\frac{1}{p^2}(1-e^{-p}) - \frac{e^{-p}}{p} \right] = \frac{k}{p^2} - \frac{k e^{-p}}{p(1-e^{-p})}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p^{3/2}}+\frac{4}{p-2}+\frac{2p-18}{p^2-9}\right\} = 2\sqrt{\frac{t}{\pi}} + 4e^{2t} + 2\cosh 3t - 6\sinh 3t$$

Ans. $\left(\because \Gamma\frac{3}{2} = \frac{\sqrt{\pi}}{2}\right)$

(iii):- $L^{-1}\left\{\frac{p^3}{p^4-a^4}\right\} = L^{-1}\left[p\left\{\frac{p}{(p^2-a^2)(p^2+a^2)}\right\}\right]$

$$= L^{-1}\left\{\frac{p}{2}\left(\frac{1}{p^2-a^2}+\frac{1}{p^2+a^2}\right)\right\} \quad \text{(using property of partial fraction)}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{p}{p^2-a^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{p}{p^2+a^2}\right\}$$

$$= \frac{1}{2}\cosh at + \frac{1}{2}\cos at$$

$$\Rightarrow L^{-1}\left\{\frac{p^3}{p^4-a^4}\right\} = \frac{1}{2}(\cosh at + \cos at)$$ Ans.

First Translation or Shifting Property :→

If $L^{-1}\{f(p)\} = F(t)$, then $\boxed{L^{-1}\{f(p-a)\} = e^{at}F(t)}$

we know that
$$f(p) = \int_0^\infty e^{-pt}F(t)\,dt \quad \text{(By definition)}$$

$$\therefore f(p-a) = \int_0^\infty e^{-(p-a)t}F(t)\,dt$$

$$\Rightarrow f(p-a) = \int_0^\infty e^{-pt}\cdot e^{at}F(t)\,dt$$

$$\Rightarrow f(p-a) = L\{e^{at}F(t)\}$$

$$\Rightarrow \boxed{L^{-1}\{f(p-a)\} = e^{at}F(t)}$$

B. Tech I Year [Subject Name: Engineering Mathematics-II]

Question No. Find the inverse Laplace transform of

(i) $\frac{15}{p^2+4p+13}$ (2015) (ii) $\frac{p+8}{p^2+4p+5}$ (2018) (iii) $\frac{p^2+2a^2}{p^4+4a^4}$

Solution (i):- $L^{-1}\left\{\frac{15}{p^2+4p+13}\right\} = L^{-1}\left\{\frac{15}{p^2+4p+4+9}\right\}$

$$= L^{-1}\left\{\frac{15}{(p+2)^2+9}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{15}{p^2+3^2}\right\} \quad \text{(using first shifting property)}$$

$$= e^{-2t}\cdot 15\,L^{-1}\left\{\frac{1}{p^2+3^2}\right\}$$

$$= e^{-2t}\frac{15}{3}\sin 3t \quad \left| L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\sin at\right.$$

$$L^{-1}\left\{\frac{15}{p^2+4p+13}\right\} = 5e^{-2t}\sin 3t$$ Ans.

(ii) :→ $L^{-1}\left\{\frac{p+8}{p^2+4p+5}\right\} = L^{-1}\left\{\frac{(p+2)+6}{(p+2)^2+1}\right\}$

$$= L^{-1}\left\{\frac{(p+2)}{(p+2)^2+1}\right\} + 6L^{-1}\left\{\frac{1}{(p+2)^2+1}\right\}$$

$$= e^{-2t}\cos t + 6\cdot e^{-2t}\sin t \quad \text{(using first shifting property)}$$

Ans.

Inverse Laplace Transform :-

If $L\{F(t)\} = f(p)$, then $F(t)$ is called the inverse laplace transform of $f(p)$ and is denoted by

$$\boxed{L^{-1}\{f(p)\} = F(t)}$$

Here L^{-1} denotes the inverse laplace transform operator.

for exp: since $L\{e^{5t}\} = \frac{1}{p-5}$

$$\therefore L^{-1}\left\{\frac{1}{p-5}\right\} = e^{5t}$$

The inverse Laplace transform given below follow at once from the results of laplace transforms given earlier:

1. $L^{-1}\left\{\frac{1}{p}\right\} = 1$
2. $L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$
3. $L^{-1}\left\{\frac{1}{p^n}\right\} = \frac{t^{n-1}}{\lfloor n-1}$, if n is a positive integer. $\left(\text{otherwise} = \frac{t^{n-1}}{\Gamma(n)}\right)$
4. $L^{-1}\left\{\frac{1}{(p-a)^n}\right\} = e^{at}\frac{t^{n-1}}{(n-1)!}$
5. $L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\sin at$
6. $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$
7. $L^{-1}\left\{\frac{1}{p^2-a^2}\right\} = \frac{1}{a}\sinh at$
8. $L^{-1}\left\{\frac{p}{p^2-a^2}\right\} = \cosh at$

Linearity Property :-

If c_1 & c_2 are constants and $L\{F_1(t)\} = f_1(p)$ and $L\{F_2(t)\} = f_2(p)$
then $L^{-1}\{c_1 f_1(p) + c_2 f_2(p)\} = c_1 L^{-1}\{f_1(p)\} + c_2 L^{-1}\{f_2(p)\}$.

Note :- The above result can be extended to more than two functions.

Question No. Find the inverse laplace transform of

(i) $\frac{3(p^2-1)^2}{2p^5}$ (ii) $\frac{1}{p^{3/2}} + \frac{4}{p-2} + \frac{2p-18}{p^2-9}$

(iii) $\frac{p^3}{p^4-a^4}$

Solution (i) :→ $\frac{3(p^2-1)^2}{2p^5} = \frac{3(p^4+1-2p^2)}{2p^5} = \frac{3p^4-6p^2+3}{2p^5}$

Now we taking inverse laplace transform

$$L^{-1}\left\{\frac{3(p^2-1)^2}{2p^5}\right\} = L^{-1}\left\{\frac{3p^4-6p^2+3}{2p^5}\right\} = \frac{3}{2}L^{-1}\left\{\frac{1}{p}\right\} - 3L^{-1}\left\{\frac{1}{p^3}\right\} + \frac{3}{2}L^{-1}\left\{\frac{1}{p^5}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{3(p^2-1)^2}{2p^5}\right\} = \frac{3}{2}(1) - 3\frac{t^2}{2!} + \frac{3}{2}\frac{t^4}{4!} \quad \left(L^{-1}\left\{\frac{1}{p^n}\right\} = \frac{t^{n-1}}{n-1!}\right)$$

$$\Rightarrow L^{-1}\left\{\frac{3(p^2-1)^2}{2p^5}\right\} = \frac{3}{2} - \frac{3}{2}t^2 + \frac{1}{16}t^4. \quad \text{Ans}$$

(ii) :→ $L^{-1}\left\{\frac{1}{p^{3/2}} + \frac{4}{p-2} + \frac{2p-18}{p^2-9}\right\}$

$$= L^{-1}\left\{\frac{1}{p^{3/2}}\right\} + 4L^{-1}\left\{\frac{1}{p-2}\right\} + 2L^{-1}\left\{\frac{p}{p^2-9}\right\} - 18L^{-1}\left\{\frac{1}{p^2-9}\right\}$$

$$= \frac{t^{1/2}}{\Gamma 3/2} + 4e^{2t} + 2\cosh 3t - \frac{18}{3}\sinh 3t$$

(iii):→ $\dfrac{p^2+2a^2}{p^4+4a^4} = \dfrac{1}{2}\left[\dfrac{1}{(p-a)^2+a^2} + \dfrac{1}{(p+a)^2+a^2}\right]$ (using Partial fraction Property)

$$\therefore L^{-1}\left\{\frac{p^2+2a^2}{p^4+4a^4}\right\} = \frac{1}{2}\cdot L^{-1}\left[\frac{1}{(p-a)^2+a^2} + \frac{1}{(p+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{a}e^{at}\sin at + \frac{1}{a}e^{-at}\sin at\right]$$

$$= \frac{1}{a}\left(\frac{e^{at}+e^{-at}}{2}\right)\sin at$$

$$L^{-1}\left\{\frac{p^2+2a^2}{p^4+4a^4}\right\} = \frac{1}{a}\sin at\cosh at \quad \text{Ans.}$$

Second Translation or Shifting Property :-

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{e^{-ap}f(p)\} = G(t), \text{ where } G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$$

Note:- we may write $G(t)$ in terms of Heaviside's unit step function as $F(t-a)\,U(t-a)$ or $F(t-a)\,H(t-a)$

∴ the above theorem can be restated as

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{e^{-ap}f(p)\} = F(t-a)\,H(t-a)$$

Question No. Evaluate

(i) $L^{-1}\left\{\dfrac{e^{-2p}}{p^2}\right\}$ (2013,22) (ii) $L^{-1}\left\{\dfrac{e^{-p}}{\sqrt{p+1}}\right\}$ (2015) (iii) $L^{-1}\left\{\dfrac{e^{-2\pi p}}{p(p^2+1)}\right\}$ (2011)

(iv) $L^{-1}\left\{\dfrac{p-1}{p^2(p-7)}\right\}$ (2011)

Solution (i):- we have

$$L^{-1}\left\{\frac{1}{p^2}\right\} = t = F(t) \quad \text{(say)}$$

$$\therefore L^{-1}\left\{e^{-2p}\cdot\frac{1}{p^2}\right\} = \begin{cases} (t-2), & t>2 \\ 0, & t<2 \end{cases}$$ (using second shifting Property)

$$L^{-1}\left\{\frac{e^{-2p}}{p^2}\right\} = (t-2)\,u(t-2) \quad \text{Ans.}$$

(ii):→ we have

$$L^{-1}\left\{\frac{1}{\sqrt{p+1}}\right\} = e^{-t}L^{-1}\left\{\frac{1}{\sqrt{p}}\right\} = \frac{1}{\sqrt{\pi t}}e^{-t}$$

$$\therefore L^{-1}\left\{\frac{e^{-p}}{\sqrt{p+1}}\right\} = \begin{cases} \dfrac{e^{-(t-1)}}{\sqrt{\pi(t-1)}}, & t>1 \\ 0, & t<1 \end{cases}$$

$$= \frac{e^{-(t-1)}}{\sqrt{\pi(t-1)}}\,u(t-1)$$

(using second shifting Property)

$L\{t^n\} = \dfrac{\Gamma(n+1)}{p^{n+1}}$

Let $n = -\frac{1}{2}$ then

$L\{t^{-1/2}\} = \dfrac{\Gamma(-\frac{1}{2}+1)}{p^{1/2}}$

$L\{t^{-1/2}\} = \dfrac{\Gamma\frac{1}{2}}{p^{1/2}}$

$\Rightarrow \dfrac{t^{-1/2}}{\Gamma\frac{1}{2}} = L^{-1}\left\{\dfrac{1}{\sqrt{p}}\right\}$

$\Rightarrow \dfrac{1}{\sqrt{\pi t}} = L^{-1}\left\{\dfrac{1}{\sqrt{p}}\right\}$

$\because \Gamma\frac{1}{2} = \sqrt{\pi}$

(iii) :→

Let

$$\frac{1}{p(p^2+1)} = \frac{A}{p} + \frac{Bp+C}{p^2+1}$$

$\Rightarrow 1 = A(p^2+1) + p(Bp+C)$

$\Rightarrow 1 = Ap^2 + A + Bp^2 + pC$

$\Rightarrow 1 = (A+B)p^2 + pC + A$

Equating coefficients of p^2, p & constant, we get

$A+B=0 \Rightarrow \boxed{B=-1}$

$\boxed{C=0}$ & $\boxed{A=1}$

$$\therefore \frac{1}{p(p^2+1)} = \frac{1}{p} - \frac{p}{p^2+1}$$

$$L^{-1}\left\{\frac{1}{p(p^2+1)}\right\} = L^{-1}\left\{\frac{1}{p} - \frac{p}{p^2+1}\right\}$$

$$= L^{-1}\left\{\frac{1}{p}\right\} - L^{-1}\left\{\frac{p}{p^2+1}\right\}$$

$$\boxed{L^{-1}\left\{\frac{1}{p(p^2+1)}\right\} = 1 - \cos t}$$ Ans.

(iv) Let $\dfrac{p-1}{p^2(p-7)} = \dfrac{A}{p} + \dfrac{B}{p^2} + \dfrac{C}{(p-7)}$

$\Rightarrow (p-1) = Ap(p-7) + B(p-7) + Cp^2$

$\Rightarrow p-1 = Ap^2 - 7Ap + Bp - 7B + Cp^2$

$\Rightarrow p-1 = (A+C)p^2 + (B-7A)p - 7B$

Comparison gives

B. Tech I Year [Subject Name: Engineering Mathematics-II]

$A+C=0$, $B-7A=1$, $-7B=-1$

$\Rightarrow A = -\frac{6}{49}$, $B = \frac{1}{7}$, $C = \frac{6}{49}$

$$\therefore L^{-1}\left\{\frac{p-1}{p^2(p-7)}\right\} = \frac{-6}{49} L^{-1}\left\{\frac{1}{p}\right\} + \frac{1}{7} L^{-1}\left\{\frac{1}{p^2}\right\} + \frac{6}{49} L^{-1}\left\{\frac{1}{p-7}\right\}$$

$$= -\frac{6}{49} + \frac{1}{7}t + \frac{6}{49}e^{7t}$$ Ans.

Change of scale property :–

If $L^{-1}\{f(p)\} = F(t)$, then

$$\boxed{L^{-1}\{f(ap)\} = \frac{1}{a}F\left(\frac{t}{a}\right)}$$

Question No: If $L^{-1}\left(\dfrac{e^{-1/p}}{\sqrt{p}}\right) = \dfrac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, find $L^{-1}\left\{\dfrac{e^{-a/p}}{\sqrt{p}}\right\}$.

Solution :→ Replacing p by kp, we get

$$L^{-1}\left(\frac{e^{-\frac{1}{kp}}}{\sqrt{kp}}\right) = \frac{1}{k} \cdot \frac{\cos 2\sqrt{\frac{t}{k}}}{\sqrt{\pi \frac{t}{k}}} = \frac{1}{\sqrt{k}} \cdot \frac{\cos 2\sqrt{\frac{t}{k}}}{\sqrt{\pi t}}$$

$$\Rightarrow L^{-1}\left(\frac{e^{-1/kp}}{\sqrt{p}}\right) = \frac{\cos 2\sqrt{\frac{t}{k}}}{\sqrt{\pi t}}$$

Putting $k = \frac{1}{a}$, we get

$$\boxed{L^{-1}\left(\frac{e^{-a/p}}{\sqrt{p}}\right) = \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}}$$ Ans.

Inverse Laplace Transform of Derivatives:-

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f^n(p)\} = L^{-1}\left[\frac{d^n}{dp^n}\{f(p)\}\right] = (-1)^n t^n F(t).$$

we have $L\{t^n F(t)\} = (-1)^n \left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n f^n(p)$

$$\therefore L^{-1}\{f^n(p)\} = (-1)^n t^n F(t).$$

Question No. Find the inverse Laplace transform of

(i) $\log\left(1+\frac{1}{p^2}\right)$ (2015, 2022), (ii) $\log\left(\frac{p+1}{p-1}\right)$ (2012)

(iii) $\cot^{-1}\left(\frac{p+3}{2}\right)$ (2013), (iv) $L^{-1}\left[\log\left(\frac{p^2+4p+5}{p^2+2p+5}\right)\right]$ (2014)

Solution (i):→ let $L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\} = F(t)$ (say)

$$\therefore L^{-1}\left[\frac{d}{dp}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}\right] = (-1)^1 t^1 F(t)$$

$$\Rightarrow L^{-1}\left[\frac{1}{\left(1+\frac{1}{p^2}\right)} \times \left(\frac{-2}{p^3}\right)\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{-2p^2}{p^3(p^2+1)}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{-2}{p(p^2+1)}\right] = -tF(t) \quad \text{——— (1)}$$

let $\dfrac{-2}{p(p^2+1)} = \dfrac{A}{p} + \dfrac{Bp+C}{p^2+1}$

$$\Rightarrow -2 = A(p^2+1) + p(Bp+C)$$

$$\Rightarrow -2 = (A+B)p^2 + Cp + A$$

Comparison gives

$\boxed{A=-2}$, $\boxed{B=2}$ & $\boxed{C=0}$

$$\therefore L^{-1}\left(\frac{1}{p} - \frac{p}{p^2+1}\right) = \frac{t}{2}F(t)$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p}\right\} - L^{-1}\left\{\frac{p}{p^2+1}\right\} = \frac{t}{2}F(t)$$

$$\Rightarrow 1 - \cos t = \frac{t}{2}F(t)$$

$$\Rightarrow \boxed{F(t) = \frac{2(1-\cos t)}{t}}$$

Ans:

(ii):→ let $L^{-1}\left\{\log\left(\frac{p+1}{p-1}\right)\right\} = F(t)$ (say)

$$\therefore L^{-1}\left[\frac{d}{dp}\{\log(p+1) - \log(p-1)\}\right] = -tF(t)$$

$$\Rightarrow L^{-1}\left[\frac{1}{(p+1)} - \frac{1}{(p-1)}\right] = -tF(t)$$

$$\Rightarrow e^{-t} - e^{t} = -tF(t)$$

$$\Rightarrow \frac{e^t - e^{-t}}{t} = F(t)$$

$$\Rightarrow \boxed{F(t) = \frac{2\sinh t}{t}} \quad \left(\because \sinh t = \frac{e^t - e^{-t}}{2}\right)$$

Ans.

(iii):→ let $L^{-1}\left\{\cot^{-1}\left(\frac{p+3}{2}\right)\right\} = F(t)$ (say)

$\therefore L^{-1}\left[\frac{d}{dp}\left\{\cot^{-1}\left(\frac{p+3}{2}\right)\right\}\right] = -tF(t)$

$\Rightarrow L^{-1}\left[\frac{-1}{1+\left(\frac{p+3}{2}\right)^2} \times \frac{1}{2}\right] = -tF(t)$ $\left(\because \frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}\right)$

$\Rightarrow L^{-1}\left[\frac{-2}{4+(p+3)^2}\right] = -tF(t)$

$\Rightarrow L^{-1}\left[\frac{-2}{(p+3)^2+2^2}\right] = -tF(t)$

$\Rightarrow -e^{-3t}\sin 2t = -tF(t)$ (using first shifting property)

$\Rightarrow \boxed{F(t) = \frac{e^{-3t}\sin 2t}{t}}$ Ans.

(iv):→ let $L^{-1}\left[\log\left(\frac{p^2+4p+5}{p^2+2p+5}\right)\right] = F(t)$ (say)

$\therefore L^{-1}\left[\frac{d}{dp}\left\{\log(p^2+4p+5) - \log(p^2+2p+5)\right\}\right] = -tF(t)$

$\Rightarrow L^{-1}\left[\frac{(2p+4)}{(p^2+4p+5)} - \frac{(2p+2)}{(p^2+2p+5)}\right] = -tF(t)$

$\Rightarrow L^{-1}\left[\frac{(2p+4)}{p^2+4p+4+1} - \frac{(2p+2)}{p^2+2p+1+4}\right] = -tF(t)$

$\Rightarrow L^{-1}\left[\frac{2p+4}{(p+2)^2+1} - \frac{2p+2}{(p+1)^2+4}\right] = -tF(t)$

$\Rightarrow L^{-1}\left[\frac{2(p+2)}{(p+2)^2+1} - \frac{2(p+1)}{(p+1)^2+2^2}\right] = -tF(t)$

$\Rightarrow 2e^{-2t}L^{-1}\left(\frac{p}{p^2+1}\right) - 2e^{-t}L^{-1}\left(\frac{p}{p^2+2^2}\right) = -tF(t)$

$\Rightarrow 2e^{-2t}\cos t - 2e^{-t}\cos 2t = -tF(t)$

$\Rightarrow F(t) = \frac{2(e^{-2t}\cos t - e^{-t}\cos 2t)}{-t}$

$\Rightarrow \boxed{F(t) = \frac{2(e^{-t}\cos 2t - e^{-2t}\cos t)}{t}}$ Ans.

Multiplication By p:-

If $L^{-1}\{f(p)\} = F(t)$ and $F(0) = 0$, then $\boxed{L^{-1}\{pf(p)\} = F'(t).}$

we have,

$L\{F'(t)\} = pf(p) - F(0) = pf(p)$ $(\because F(0) = 0)$

$\therefore \boxed{L^{-1}\{pf(p)\} = F'(t)}$

Note: 1. If $F(0) \neq 0$, then

$L^{-1}\{pf(p) - F(0)\} = F'(t)$

or

$L^{-1}\{pf(p)\} = F'(t) + F(0)\delta(t)$

where $\delta(t)$ is the unit impulse function

2. Generalizations to $L^{-1}\{p^n f(p)\}$ are also possible for $n = 2, 3, \ldots$.

Division By p :-

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(u)\,du$$

Also, $L^{-1}\left\{\frac{f(p)}{p^2}\right\} = \int_0^t\int_0^t F(u)\,du\,du$

- - - - - - -

$$L^{-1}\left\{\frac{f(p)}{p^n}\right\} = \int_0^t\int_0^t \cdots \int_0^t F(u)\,du\,du \cdots du \quad (n\text{-times})$$

(n times)

Question No: Find the inverse laplace transform of

(i) $\frac{1}{p(p+1)^3}$ (ii) $\frac{1}{p\sqrt{p+4}}$ (iii) $\frac{1}{p(p^2+a^2)}$

Solution (i) :- Since $L^{-1}\left[\frac{1}{(p+1)^3}\right] = \frac{e^{-t}t^2}{2!}$ (By first shifting property)

$$\therefore L^{-1}\left[\frac{1}{p(p+1)^3}\right] = \frac{1}{2}\int_0^t t^2 e^{-t}\,dt$$

$$= \frac{1}{2}\left[t^2(-e^{-t}) - 2te^{-t} + 2(-e^{-t})\right]_0^t$$

$$= \frac{1}{2}\left[(-t^2-2t-2)e^{-t}+2\right]$$

$$\boxed{L^{-1}\left[\frac{1}{p(p+1)^3}\right] = 1 - e^{-t}\left(1+t+\frac{t^2}{2}\right)}$$

Ans.

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(ii) :- $L^{-1}\left(\frac{1}{\sqrt{p+4}}\right) = e^{-4t}L^{-1}\left(\frac{1}{\sqrt{p}}\right) = e^{-4t}\cdot\frac{t^{-1/2}}{\Gamma 1/2} = \frac{e^{-4t}}{\sqrt{\pi t}}$ $\quad \because \Gamma 1/2 = \sqrt{\pi}$

$$\therefore L^{-1}\left(\frac{1}{p\sqrt{p+4}}\right) = \int_0^t \frac{e^{-4u}}{\sqrt{\pi u}}\,du$$

$$= \frac{1}{\sqrt{\pi}}\int_0^t e^{-4u}u^{-1/2}\,du$$

Put $2\sqrt{u} = x$ $\therefore u^{-1/2}du = dx$

$$= \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-x^2}\,dx$$

$$= \frac{1}{2}\left(\frac{2}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-x^2}\,dx\right)$$

$$\therefore \boxed{L^{-1}\left(\frac{1}{p\sqrt{p+4}}\right) = \frac{1}{2}\,\mathrm{erf}(2\sqrt{t})}$$

Ans.

The error function :-
The error function denoted by $\mathrm{erf}\sqrt{t}$ is defined as
$$\mathrm{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-\eta^2}d\eta$$

Note :- $\lim_{x\to 0} \mathrm{erf}\,x = 0$ and $\lim_{x\to\infty} \mathrm{erf}\,x = 1$

(iii) $L^{-1}\left(\frac{1}{p^2+a^2}\right) = \frac{1}{a}\sin at$

$$\therefore L^{-1}\left(\frac{1}{p(p^2+a^2)}\right) = \int_0^t \frac{1}{a}\sin au\,du$$

$$= \left[-\frac{1}{a^2}\cos au\right]_0^t$$

$$\therefore \boxed{L^{-1}\left(\frac{1}{p(p^2+a^2)}\right) = \frac{1}{a^2}\left[1-\cos at\right]}$$

Ans.

Convolution Theorem :→ (2016, 2018)

If $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$, then

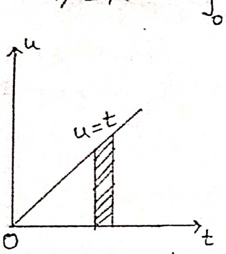
$$L^{-1}\{f(p)\,g(p)\} = F*G = \int_0^t F(u)\,G(t-u)\,du$$

Proof:- let $\phi(t) = \int_0^t F(u)\,G(t-u)\,du$

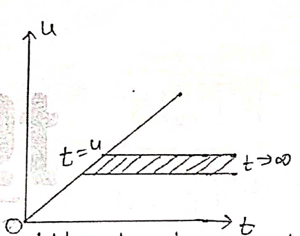
Then

$$L\{\phi(t)\} = \int_0^\infty e^{-pt}\left[\int_0^t F(u)\,G(t-u)\,du\right]dt$$

$$\Rightarrow L\,\phi(t) = \int_0^\infty \int_0^t e^{-pt}F(u)\,G(t-u)\,du\,dt$$



Before changing the order of integration



After changing the order of Integration

we get, $L\,\phi(t) = \int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-pt}F(u)\,G(t-u)\,dt\,du$

$$= \int_0^\infty e^{-pu}F(u)\left[\int_u^\infty e^{-p(t-u)}G(t-u)\,dt\right]du$$

$$= \int_0^\infty e^{-pu}F(u)\left[\int_0^\infty e^{-pv}G(v)\,dv\right]du$$

on putting $t-u=v$

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$$= \int_0^\infty e^{-pu}F(u)\,g(p)\,du = g(p) = \int_0^\infty e^{-pu}F(u)\,du$$

$$= g(p)\cdot f(p) = f(p)\,g(p)$$

$$\Rightarrow L^{-1}\{f(p)\,g(p)\} = \phi(t) = \int_0^t F(u)\,G(t-u)\,du.$$

we call $F*G$, the convolution of F and G & the theorem is called the convolution theorem or the convolution property.

Question No:→ Use convolution theorem to evaluate

(i) $L^{-1}\left\{\frac{p}{(p^2+1)(p^2+4)}\right\}$ (2013), (ii) $L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\}$ (2010)

(iii) $L^{-1}\left\{\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right\}$ (2018)

Solution:- (i) If $L^{-1}\{f(p)\} = F(t)$ & $L^{-1}\{g(p)\} = G(t)$, then

$$L^{-1}\{f(p)\cdot g(p)\} = F*G = \int_0^t F(u)\cdot G(t-u)\,du$$

let $f(p) = \frac{1}{p^2+4}$ and $g(p) = \frac{p}{p^2+1}$

$\therefore F(t) = \frac{1}{2}\sin 2t$ and $G(t) = \cos t$

$\therefore F(u) = \frac{1}{2}\sin 2u$ and $G(t-u) = \cos(t-u)$

$$\therefore L^{-1}\left\{\frac{p}{(p^2+1)(p^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2u\cdot\cos(t-u)\,du$$

$$= \frac{1}{4}\int_0^t 2\sin 2u\cdot\cos(t-u)\,du$$

Note:- $\left.\begin{array}{l}\sin(A+B) = \sin A\cos B + \cos A\sin B\\ \sin(A-B) = \sin A\cos B - \cos A\sin B\end{array}\right] \Rightarrow \sin(A+B)+\sin(A-B) = 2\sin A\cos B$

$$= \frac{1}{4}\int_0^t [\sin(u+t) + \sin(3u-t)]\,du$$

$$= \frac{1}{4}\left[-\cos(u+t) - \frac{\cos(3u-t)}{3}\right]_0^t$$

$$= \frac{1}{4}\left[\left(-\cos 2t - \frac{\cos 2t}{3}\right) - \left(-\cos t - \frac{\cos t}{3}\right)\right]$$

$$= \frac{1}{4}\left[-\frac{4}{3}\cos 2t + \frac{4}{3}\cos t\right]$$

$$L^{-1}\left\{\frac{p}{(p^2+1)(p^2+4)}\right\} = \frac{1}{3}(\cos t - \cos 2t) \quad \text{Ans.}$$

(ii):- $\dfrac{p}{(p^2+4)^2} = \dfrac{1}{(p^2+4)} \cdot \dfrac{p}{(p^2+4)}$

Let $f(p) = \dfrac{1}{p^2+4}$ and $g(p) = \dfrac{p}{(p^2+4)}$

$\therefore F(t) = L^{-1}\{f(p)\} = L^{-1}\left\{\dfrac{1}{p^2+4}\right\} = \dfrac{1}{2}\sin 2t$

and $G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\dfrac{p}{p^2+4}\right\} = \cos 2t$

Now $F(u) = \frac{1}{2}\sin 2u$, $G(t-u) = \cos 2(t-u)$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\} = \int_0^t \frac{1}{2}\sin 2u \cdot \cos 2(t-u)\,du$$

$$= \frac{1}{4}\int_0^t 2\sin 2u \cdot \cos 2(t-u)\,du$$

$$= \frac{1}{4}\int_0^t [\sin 2t + \sin(4u-2t)]\,du$$

$$= \frac{1}{4}\left[u\sin 2t - \frac{\cos(4u-2t)}{4}\right]_0^t$$

$$= \frac{t}{4}\sin 2t \quad \text{Ans.}$$

(iii):→ $\dfrac{p^2}{(p^2+a^2)(p^2+b^2)} = \dfrac{p}{(p^2+a^2)} \cdot \dfrac{p}{(p^2+b^2)}$

Let $f(p) = \dfrac{p}{(p^2+a^2)}$ and $g(p) = \dfrac{p}{(p^2+b^2)}$

$\therefore F(t) = L^{-1}\{f(p)\} = L^{-1}\left\{\dfrac{p}{p^2+a^2}\right\} = \cos at$

and $G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\dfrac{p}{p^2+b^2}\right\} = \cos bt$

Now, $F(u) = \cos au$, $G(t-u) = \cos b(t-u)$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right\} = \int_0^t \cos au \cdot \cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t 2\cos au \cdot \cos b(t-u)\,du$$

Note:- $\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\cos(A-B) = \cos A\cos B + \sin A\sin B$ $\Rightarrow \cos(A+B) + \cos(A-B) = 2\cos A\cos B$

$$= \frac{1}{2}\int_0^t [\cos\{(a-b)u + bt\} + \cos\{(a+b)u - bt\}]\,du$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)u+bt\}}{(a-b)} + \frac{\sin\{(a+b)u-bt\}}{(a+b)}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{(a-b)} + \frac{\sin at + \sin bt}{(a+b)}\right]$$

$$= \frac{a\sin at - b\sin bt}{(a^2-b^2)} \quad \text{Ans.}$$

Applications to Differential Equations

1. Solution of Ordinary differential equations with constant coefficients

The advantage of using Laplace transform method is that it yields the particular solution directly without the necessity of first finding the general solution and then evaluating the arbitrary constant.

Steps to solve Ode:

1. Take Laplace transform of both sides of the given differential equation, using initial conditions. This gives an algebraic equation.
2. Solve the algebraic equation to get $\bar{y}$ in terms of p.
3. Take inverse Laplace transform of both sides. This gives y as a function of t which is the desired solution.

Remember: $L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - \dots - pF^{(n-2)}(0) - F^{(n-1)}(0)$

Example: 1 Using Laplace transform, find the solution of the initial value problem:

$$\frac{d^2y}{dt^2} + 9y = 6\cos 3t \, ; \; y(0) = 2, \; y'(0) = 0. \qquad [2015]$$

Sol. The given differential equation is:

$$y'' + 9y = 6\cos 3t \qquad \text{—— (1)}$$

Taking Laplace transform on both sides of eqn.(1), we get

$$L(y'') + 9L(y) = 6L(\cos 3t)$$

$$\Rightarrow [p^2\bar{y} - py(0) - y'(0)] + 9\bar{y} = \frac{6p}{p^2+9} \qquad \text{Here, } \bar{y} = L(y)$$

$$\Rightarrow (p^2+9)\bar{y} - 2p = \frac{6p}{p^2+9}$$

$$\Rightarrow \bar{y} = \frac{6p}{(p^2+9)^2} + \frac{2p}{p^2+9} \qquad \text{—— (2)}$$

Taking inverse Laplace transform on both sides of (2), we get

$$\boxed{y(t) = t\sin t + 2\cos 3t} \qquad \because L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \frac{t}{2a}\sin at$$

Example: 2 Solve the following differential equation using Laplace transform:

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2e^t$$

where $y(0) = 1, \; \left(\frac{dy}{dt}\right)_{t=0} = 0, \; \left(\frac{d^2y}{dt^2}\right)_{t=0} = -2.$ [2018]

Sol. The given equation is:

$$y''' - 3y'' + 3y' - y = t^2e^t \qquad \text{—— (1)}$$

Taking Laplace transform on both sides of eqn.(1), we get

$$L(y''') - 3L(y'') + 3L(y') - L(y) = L(t^2e^t)$$

$$\Rightarrow [p^3\bar{y} - p^2y(0) - py'(0) - y''(0)] - 3[p^2\bar{y} - py(0) - y'(0)] + 3[p\bar{y} - y(0)] - \bar{y} = \frac{2}{(p-1)^3}, \quad \bar{y} = L(y)$$

$$\Rightarrow [p^3\bar{y} - p^2 + 2] - 3[p^2\bar{y} - p] + 3[p\bar{y} - 1] - \bar{y} = \frac{2}{(p-1)^3}$$

$$\Rightarrow [P^3-3P^2+3P-1]\bar{y} - P^2+3P-1 = \frac{2}{(P-1)^3}$$

$$\Rightarrow [P-1]^3\bar{y} - P^2+3P-1 = \frac{2}{(P-1)^3}$$

$$\Rightarrow [P-1]^3\bar{y} = P^2-3P+1+\frac{2}{(P-1)^3}$$

$$\Rightarrow \bar{y} = \frac{(P-1)^2}{(P-1)^3} - \frac{P}{(P-1)^3} + \frac{2}{(P-1)^6}$$

$$= \frac{1}{P-1} - \frac{(P-1)+1}{(P-1)^3} + \frac{2}{(P-1)^6}$$

$$= \frac{1}{P-1} - \frac{1}{(P-1)^2} - \frac{1}{(P-1)^3} + \frac{2}{(P-1)^6} \quad \text{——— (2)}$$

Taking inverse Laplace transform on both sides of eqn (2), we get

$$y = e^t - te^t - \frac{t^2}{2}e^t + \frac{t^5e^t}{60}$$

$$\therefore \boxed{y = \left(1 - t - \frac{t^2}{2} + \frac{t^5}{60}\right)e^t.}$$

Example: 3 Solve by Laplace transform:

$$\frac{d^2y}{dt^2} + y = t\cos 2t, \; t>0$$

given that $y = \frac{dy}{dt} = 0$ for $t=0$. [2016]

Sol. The given equation is:

$$y'' + y = t\cos 2t \quad \text{——— (1)}$$

Taking Laplace transform on both sides of eqn (1), we get

$$L(y'') + L(y) = L(t\cos 2t)$$

$$\Rightarrow [P^2\bar{y} - Py(0) - y'(0)] + \bar{y} = -\frac{d}{dP}\left(\frac{P^2}{P^2+4}\right), \quad \text{where } \bar{y} = L(y)$$

$$\Rightarrow (P^2+1)\bar{y} = \frac{P^2-4}{(P^2+4)^2}$$

$$\Rightarrow \bar{y} = \frac{P^2-4}{(P^2+1)(P^2+4)^2}$$

$$\Rightarrow \bar{y} = -\frac{5}{9}\cdot\frac{1}{P^2+1} + \frac{5}{9}\cdot\frac{1}{P^2+4} + \frac{8}{3}\cdot\frac{1}{(P^2+4)^2} \quad \text{——— (2)}$$

Taking inverse Laplace transform on both sides of eqn (2), we get

$$y = -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{8}{3}\cdot\frac{1}{16}[\sin 2t - 2t\cos 2t]$$

$$\left[\because L^{-1}\left\{\frac{1}{(P^2+a^2)^2}\right\} = \frac{1}{2a^3}[\sin at - at\cos at]\right]$$

$$\therefore \boxed{y = -\frac{5}{9}\sin t + \frac{4}{9}\sin 2t - \frac{t}{3}\cos 2t.}$$

Some Practice Problems:

Example: 4 Solve by using Laplace transform:

$y''+2y'+y = te^{-t}$, $y(0)=1$, $y'(0)=-2$. [2015]

Example: 5 $(D^3-D^2-D+1)y = 8te^{-t}$

$y(0)=0$, $y'(0)=1$, $y''(0)=0$. [2014]

Sol. Proceed as in above examples.

2. Solution of simultaneous ordinary differential equations:

Laplace transform technique can also be used in solving two or more simultaneous ordinary differential equations.

This process is illustrated as follows:

Examples 1 Solve the simultaneous equations:

$$\frac{dx}{dt} - y = e^t \quad , \quad \frac{dy}{dt} + x = \sin t$$

given that $x(0) = 1$, $y(0) = 0$. [2010-2011]

Sol. Taking Laplace transform of the given equations, we get

$$[p\bar{x} - x(0)] - \bar{y} = \frac{1}{p-1} \qquad \text{where } \bar{x} = L(x),\ \bar{y} = L(y)$$

i.e., $p\bar{x} - 1 - \bar{y} = \frac{1}{p-1}$ $[\because x(0) = 1]$

i.e., $p\bar{x} - \bar{y} = \frac{p}{p-1}$ —— ①

and $[p\bar{y} - y(0)] + \bar{x} = \frac{1}{p^2+1}$

i.e., $\bar{x} + p\bar{y} = \frac{1}{p^2+1}$ —— ② $[\because y(0) = 0]$

Solving ① and ② for $\bar{x}$ and $\bar{y}$, we have

$$\bar{x} = \frac{p^2}{(p-1)(p^2+1)} + \frac{1}{(p^2+1)^2}$$

$$\bar{x} = \frac{1}{2}\left[\frac{1}{p-1} + \frac{p}{p^2+1} + \frac{1}{p^2+1}\right] + \frac{1}{(p^2+1)^2}$$

and $\bar{y} = \frac{p}{(p^2+1)^2} - \frac{p}{(p-1)(p^2+1)}$

$$\therefore \bar{y} = \frac{p}{(p^2+1)^2} - \frac{1}{2}\left[\frac{1}{p-1} - \frac{p}{p^2+1} + \frac{1}{p^2+1}\right]$$

Taking Inverse Laplace transform on both sides, we get

$$x = \frac{1}{2}L^{-1}\left[\frac{1}{p-1} + \frac{p}{p^2+1} + \frac{1}{p^2+1}\right] + L^{-1}\left[\frac{1}{(p^2+1)^2}\right]$$

$$= \frac{1}{2}[e^t + \cos t + \sin t] + \frac{1}{2}[\sin t - t\cos t]$$

$$\left[\because L^{-1}\left\{\frac{1}{(p^2+a^2)^2}\right\} = \frac{1}{2a^3}[\sin at - at\cos at]\right]$$

$$\therefore \boxed{x = \frac{1}{2}[e^t + \cos t + 2\sin t - t\sin t]}$$

$$y = L^{-1}\left[\frac{p}{(p^2+1)^2}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{p-1} - \frac{p}{p^2+1} + \frac{1}{p^2+1}\right]$$

$$= \frac{1}{2}t\sin t - \frac{1}{2}[e^t - \cos t + \sin t]$$

$$\left[\because L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at\right]$$

$$\therefore \boxed{y = \frac{1}{2}[t\sin t - e^t + \cos t - \sin t]}.$$

Examples 2 Solve the simultaneous equations:

$$\frac{d^2x}{dt^2} + 5\frac{dy}{dt} - x = t, \quad 2\frac{dx}{dt} - \frac{d^2y}{dt^2} + 4y = 2$$

given that when $t=0$, $x=0$, $y=0$, $\frac{dx}{dt}=0$, $\frac{dy}{dt}=0$. [2022]

Sol. Let $L\{x(t)\} = \bar{x}(p)$ and $L\{y(t)\} = \bar{y}(p)$

then, taking Laplace transform of given equations, we get

$$\{p^2\bar{x} - px(0) - x'(0)\} + 5\{p\bar{y} - y(0)\} - \bar{x} = \frac{1}{p^2}$$

and $$2\{p\bar{x} - x(0)\} - \{p^2\bar{y} - py(0) - y'(0)\} + 4\bar{y} = \frac{2}{p}$$

Using the given initial conditions, these equations reduce to

$$(p^2-1)\bar{x} + 5p\bar{y} = \frac{1}{p^2} \quad \text{—— ①}$$

and $$2p\bar{x} - (p^2-4)\bar{y} = \frac{2}{p} \quad \text{—— ②}$$

Eliminating $\bar{y}$ between ① and ②, we find that

$$\{(p^2-1)(p^2-4) + 10p^2\}\bar{x} = \frac{p^2-4}{p^2} + 10$$

$$\therefore \quad \bar{x} = \frac{11p^2 - 4}{p^2(p^2+1)(p^2+4)}$$

$$= \frac{-1}{p^2} + \frac{5}{p^2+1} - \frac{4}{p^2+4}$$

Taking inverse Laplace transform, we get

$$\boxed{x = -t + 5\sin t - 2\sin 2t} \quad \text{—— ③}$$

Again eliminating $\bar{x}$ between ① and ②, we get

$$\{10p^2 + (p^2-1)(p^2-4)\}\bar{y} = \frac{2}{p} - \frac{2(p^2-1)}{p}$$

$$\therefore \quad \bar{y} = \frac{4-2p^2}{p(p^2+1)(p^2+4)}$$

$$= \frac{1}{p} - \frac{2p}{p^2+1} + \frac{p}{p^2+4}$$

Taking inverse Laplace transform, we get

$$\boxed{y = 1 - 2\cos t + \cos 2t} \quad \text{—— ④}$$

Thus ③ and ④ together constitute the desired solution.

Examples 3 Solve the following simultaneous equations by using Laplace transform:

$$\frac{dx}{dt} + \frac{dy}{dt} + x + y = 1, \quad \frac{dy}{dt} = 2x + y;$$

$$x(0) = 0, \quad y(0) = 1.$$ [2011]

Sol. Proceed as in above examples.

Examples 4 The co-ordinates (x,y) of a particle moving along a plane curve at any time t are given by $\frac{dy}{dt}+2x=\sin 2t$, $\frac{dx}{dt}-2y=\cos 2t$; $(t>0)$. It is given that at $t=0$, $x=1$ and $y=0$. Show using transforms that the particle moves along the curve $4x^2+4xy+5y^2=4$. [2017]

Sol. The given equations are

$$\frac{dy}{dt}+2x=\sin 2t \quad \text{——} ①$$

$$\frac{dx}{dt}-2y=\cos 2t \quad \text{——} ②$$

Above equation may be rewritten as

$$2x+Dy=\sin 2t$$

$$Dx-2y=\cos 2t, \qquad \text{where } D\equiv \frac{d}{dt}$$

Taking Laplace transform of eqn. ① on both sides, we get

$$2\bar{x}+p\bar{y}-y(0)=\frac{2}{p^2+4}, \quad \text{where } \bar{x}=L(x),\ \bar{y}=L(y)$$

$$\Rightarrow \quad 2\bar{x}+p\bar{y}=\frac{2}{p^2+4} \quad \text{——} ③ \quad [\because y(0)=0]$$

Again, taking Laplace transform of eqn. ② on both sides, we get

$$p\bar{x}-x(0)-2\bar{y}=\frac{p}{p^2+4}$$

$$\Rightarrow \quad p\bar{x}-2\bar{y}=\frac{p}{p^2+4}+1 \quad \text{——} ④ \quad [\because x(0)=0]$$

Multiplying eqn. ③ by 2 and eqn. ④ by p and then adding, we get

$$4\bar{x}+p^2\bar{x}=\frac{4}{p^2+4}+\frac{p^2}{p^2+4}+p$$

$$\Rightarrow \quad (4+p^2)\bar{x}=1+p$$

$$\therefore \quad \bar{x}=\frac{1+p}{4+p^2}=\frac{1}{4+p^2}+\frac{p}{p^2+4}$$

Taking inverse Laplace transform, we get

$$\boxed{x=\frac{1}{2}\sin 2t+\cos 2t} \quad \text{——} ⑤$$

Again, multiplying eqn. ③ by p and eqn. ④ by -2, then adding, we get,

$$p^2\bar{y}+4\bar{y}=\frac{2p}{p^2+4}-\frac{2p}{p^2+4}-2$$

$$\Rightarrow \quad \bar{y}=\frac{-2}{p^2+4} \quad \text{——}$$

Taking inverse Laplace transform, we get $y=-\sin 2t$.

Now, $4x^2=4\left[\frac{1}{4}\sin^2 2t+\cos^2 2t+\sin 2t\cos 2t\right]$

$5y^2 = 5\sin^2 2t$

$4xy = 4\left[\left(\frac{1}{2}\sin 2t + \cos 2t\right)\cdot(-\sin 2t)\right]$

$= -(2\sin^2 2t + 4\sin 2t\cos 2t)$

$\therefore\ 4x^2 + 5y^2 + 4xy = 4\sin^2 2t + 4\cos^2 2t = 4.$

Hence, the result.

5 Year's

University Paper Questions

(AKTU Question Bank)

10 Years AKTU University Examination Questions		Unit-2
CO2:Apply the concept of Laplace Transform to solve differential equations		(CO 2)
S. No	Questions	Session
	LAPLACE TRANSFORM	
1	Find theLaplace transformation of $f(t) = cosh at. cos at$	
2	Explain first shifting property of Laplace transform.	[2010,2011,2013] [Short]
3	Explain second translation property of Laplace transform.	[2006,2008,2009,2012] [Short]
4	Explain change of scale property of Laplace transform.	[2008][Short]
5	(i) If Laplace transform of f (t) is F (p), then show that Laplace transform of $e^{at}f(t)$ is F (p-a) , where a is any real number. (ii)Prove that $L\left[e^{at}f(t)\right] = F(p-a)$.	[2013,2017]
6	Find the Laplace transform of $F(t)=\begin{cases} t^2, & 0<t<2 \\ t-1, & 2<t<3 \\ 7, & t>3 \end{cases}$	[2007] [Short]
7	If $L(\cos^2 at) = \dfrac{p^2+2}{p(p^2+4)}$, find $L\left(\cos^2 at\right)$.	[2006] [Short]
8	Find the laplace transform of $F(t)=\begin{cases} \sin t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$.	[2009] [Short]
9	Find the laplace transform of $F(t)=\begin{cases} \dfrac{t}{w}, & 0<t<w \\ 1-\dfrac{t}{w}, & w<t<2w \\ 1, & 2w<t<\infty \end{cases}$.	[2010] [Short]
10	Find Laplace of $F(t)=e^{-at}\cos\beta t$.	[2010] [Long]
11	Find Laplace of $F(t)=\cosh at\ \cos bt$.	[2016] [Short]
12	Find $L(e^{2t+3})$.	[2011]
13	Evaluate: $L(e^{-t}\cos t)$.	[2022]
14	Find $L(t^3e^{-3t})$.	[2010]

Engineering Mathematics-II (BAS-203) Question Bank

15	Find the laplace transform of $f(t) = t\sin\sqrt{7}\,t$.	[2013]
16	Find the Laplace transform of $f(t) = \dfrac{\sin at}{t}$	[2017]
17	Find the Laplace transform of $\dfrac{1-\cos t}{t}$.	[2015] [Short]
18	Find the Laplace transform of $\dfrac{\cos at-\cos bt}{t}$.	[2017] [Long]
19	Find the Laplace transform of $te^{-t}\cosh t$.	[2014] [Long]
20	Find the Laplace transform of $\dfrac{e^{-t}\sin t}{t}$.	[2012] [Long]
21	Evaluate $\int_0^\infty \dfrac{e^{-t}\sin^2 t}{t}\,dt$.	[2008,2010,2011] [Short]
22	Find Laplace of $te^{-t}\sin 2t$.	[2010] [Long]
23	Find Laplace of $t^2e^t\sin 4t$.	[2011] [Short]
24	Prove that $\int_{t=0}^{\infty}\int_{u=0}^{t} e^{-t}\dfrac{\sin u}{u}\,du\,dt = \dfrac{\pi}{4}$.	[2011] [Short]
25	Using Laplace transform, Evaluate $\int_0^\infty \dfrac{1}{u}e^{-4u}\sin 3u\ du$.	[2013] [Short]
26	Write the Laplace transform of Unit step function.	[2014] [Short]
27	Find L(F(t)), where F(t) is defined by F(t)={sin t, $0<t<\pi$; 0, $\pi<t<2\pi$ and F(t+2π)=F(t)	[2013]
28	F(t)={t, $0\le t\le 1$; 1, $1\le t\le 2$ Where F(t) has period 2.	[2013]
29	Find the Laplace Transform of "saw-tooth wave" function f(t) which is periodic with period 1 and defined as f(t)= kt in 0< t <1.	[2017]
30	Find the Laplace Transform of $\sin 2t\ u(t-\pi)$	[2014]
31	Draw the graph and find the Laplace Transform of the triangular wave function of period 2π given by F(t)= t, $0<t\le\pi$; $2\pi-t$, $\pi<t<2\pi$	[2018]

32	Draw the graph and find the Laplace Transform of the following function of period 2a: $f(t) = \begin{cases} \frac{h}{a}t, & 0 < t < a \\ \frac{h}{a}(2a - t), & a < t < 2a \end{cases}$	[2011,22]
	INVERSE LAPLACE TRANSFORM	
35	Find Inverse Laplace Transform of $\frac{1}{(s-2)^{3/2}}$.	
36	Find the function whose Laplace transform is $\frac{e^{-\pi p}}{p^2+2}$.	[2012]
37	Find the function whose laplace transform is $F(P) = \frac{8}{P^2 - P - 2}$.	[2013]
38	Find : $L^{-1}\left(\frac{e^{-\pi p}}{p^2}\right)$	[2015]
39	Evaluate : $L^{-1}\left(\frac{e^{-2p}}{p^2}\right)$	[2022]
40	Find : $L^{-1}\left(\frac{1}{p^{3/2}}\right)$.	[2011]
41	Find : $L^{-1}\left(\frac{1}{p^2-3p+3}\right)$	[2013]
42	Find $L^{-1}\left[\frac{3}{P^2+2P-8}\right]$.	[2014]
43	Find $L^{-1}\left[\frac{2}{(p-1)(p-2)}\right]$.	[2011]
44	Find $L^{-1}\left(\frac{1}{\sqrt{p}}\right)$	[2011]
45	Find inverse Laplace transform of the function $f(p) = \frac{p}{2p^2+8}$.	[2016]
45	Find the Inverse Laplace Transform of $\frac{s+1}{s^2-6s+25}$.	[2015]
47	Find the Inverse Laplace Transform of $\frac{s+8}{s^2+4s+5}$	[2018]
48	Evaluate: $$L^{-1}\left(\frac{e^{-s}}{\sqrt{s+1}}\right)$$	[2015]

49	$L^{-1}\left(\frac{e^{-2\pi s}}{s(s^2+1)}\right)$	[2011]
50	Find the Inverse Laplace Transform of $\frac{s-1}{s^2(s-7)}$	[2011]
51	Find the Inverse Laplace Transform of $log\left(1+\frac{1}{s^2}\right)$	[2015,22]
52	State and Prove Convolution Theorem.	[2016-18]
53	Find the Inverse Laplace Transform of $log\left(\frac{s+1}{s-1}\right)$	[2012]
54	Find the Inverse Laplace Transform of $\left(\frac{s+3}{2}\right)$	[2013]
55	Find the Inverse Laplace Transform of $log\left(\frac{s^2+4s+5}{s^2+2s+5}\right)$	[2014]
56	Use convolution theorem to evaluate: $$L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\},$$	[A.K.T.U,2018]
57	Prove that: $L^{-1}\left\{\frac{1}{(p^2+1)^3}\right\} = \frac{1}{8}[(3-t^2)\sin t - 3t\cos t]$.	[A.K.T.U,2016]
58	Evaluate: $L^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\}$	(A.K.T.U.2017)
59	Find the inverse Laplace transformation of $$\log\left\{\frac{p^2+1}{p(p+1)}\right\}$$	(U.P.T.U.2014)
60	State Convolution Theorem and hence evaluate $$L^{-1}\left(\frac{s}{(s^2+4)(s^2+1)}\right)$$	[2013]
61	Using convolution theorem, prove that $$L^{-1}\left[\frac{1}{p^3(p^2+1)}\right] = \frac{t^2}{2} + \cos t - 1$$	[G.B.T.U 2012]
62	Using convolution theorem, prove that $$L^{-1}\left[\frac{1}{(p^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$$	(U.P.T.U.2015)
63	Use convolution theorem to find $$L^{-1}\left[\frac{1}{(p^2+4)(p+2)}\right]$$	(A.K.T.U.2016)

64	Use convolution theorem to find $L^{-1}\left[\frac{16}{(p-2)(p+2)^2}\right]$	(U.P.T.U.2014)
65	Use convolution theorem to find $L^{-1}\left[\frac{p}{(p^2+a^2)^3}\right]$	(M.T.U. 2012)
66	Use convolution theorem to find $L^{-1}\left[\frac{1}{p^2(p+1)^2}\right]$	(G.B.T.U.2013)
67	Use convolution theorem to evaluate $L^{-1}\left\{\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right\}$	[U.P.T.U 2014]
	Application of LAPLACE TRANSFORM	
68	Using Laplace Transform to solve the following differential equation: $\frac{d^2x}{dt^2}+9x=cos2t$, if $x(0)=1, x\left(\frac{\pi}{2}\right)=-1.$	
69	$\frac{d^2x}{dt^2}+16x=2\sin 4t; x(0)=-\frac{1}{2}, x'(0)=0.$	[2014]
70	Solve the following differential equation using Laplace transform $\frac{d^3y}{dt^3}-3\frac{d^2y}{dt^2}+3\frac{dy}{dt}-y=t^2e^t$ Where $y(0)=1, \left(\frac{dy}{dt}\right)_{t=0}=0, \left(\frac{d^2y}{dt^2}\right)_{t=0}=-2.$	[A.K.T.U,2018]
71	Solve by Laplace transform: $\frac{d^2y}{dt^2}+y=t\cos 2t, t>0$ given that $y=\frac{dy}{dt}=0$ for $t=0$	[A.K.T.U,2016]
72	Solve the simultaneous equations: $\frac{dx}{dt}-y=e^t, \frac{dy}{dt}+x=\sin t$, given $x(0)=1, y(0)=0.$	[U.K.T.U,2011 G.B.T.U]
73	Solve the simultaneous equations $\frac{d^2x}{dt^2}+5\frac{dy}{dt}-x=t,\quad 2\frac{dx}{dt}-\frac{d^2y}{dt^2}+4y=2$ Given that when $t=0, x=0, y=0, \frac{dx}{dt}=0, \frac{dy}{dt}=0.$	[A.K.T.U 2022]
74	The co-ordinates (x,y) of a particle moving along a plane curve at any time time given by $\frac{dy}{dt}+2x=\sin 2t, \frac{dx}{dt}-2y=\cos 2t; (t>0)$. It is given that at	[A.K.T.U 2017]

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	$t=0, x=1$ and $y=0$. Show using transforms that the particle moves along the curve $4x^2+4xy+5y^2=4$.	
75	Use Laplace transform to solve: $\frac{dx}{dt}+y=\sin t, \frac{dy}{dt}+x=\cos t$ given thet $x=2, y=0$ at $t=0$.	[G.B.T.U 2012]
76	Solve the following simultaneous DE's by Laplace transform $3\frac{dx}{dt}-y=2t, \frac{dx}{dt}+\frac{dy}{dt}-y=0$ with the conditions $x(0)=y(0)=0.$	[A.K.T.U 2022]
77	Solve the following DE's by Laplace transform $\frac{d^2x}{dt^2}+9x=\sin 2t, x(0)=1, x'(0)=0.$	[G.B.T.U 2013]
78	Solve the following DE's by Laplace transform $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+8y=\sin x, y(0)=1, y'(0)=0$	[G.B.T.U 2013]
79	Solve the following DE's by Laplace transform $\frac{d^2y}{dx^2}+n^2y=a\sin(nx+2)$, given: $y(0)=0$ and $y'(0)=0.$	[G.B.T.U 2010]
80	Solve the following DE's by Laplace transform $\frac{d^2y}{dt^2}+9y=\sin 3t$, given: $y=0, \frac{dy}{dt}=0$ at $t=0$	[M.T.U 2012]
81	Using Laplace transformation, solve the differential equation $\frac{d^2x}{dt^2}+9x=\cos 2t$, if $x(0)=1, x\left(\frac{\pi}{2}\right)=-1$	(A.K.T.U.2017)
82	Solve the following DE's by Laplace transform $\frac{d^2y}{dt^2}-2\frac{dy}{dt}+y=e^t, y(0)=0 y'(0)=1$	[M.T.U 2013]
83	Solve the following DE's by Laplace transform $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+8x=e^{-3t}-e^{-5t}; x(0)=0, x'(0)=0$	[U.P.T.U 2015]
84	Solve the following DE's by Laplace transform $y''+2y'+y=te^{-t}; y(0)=1, y'(0)=-2.$	[G.B.T.U 2012]
85	Solve the following DE's by Laplace transform $(D^3-D^2-D+1)y=8te^{-t}; y(0)=0, y'(0)=1, y''(0)=0$	[U.P.T.U 2014]

86	Using Laplace transform, find the solution of the initial value problem $$\frac{d^2y}{dt^2}+9y=6\cos 3t; y(0)=2, y'(0)=0$$	(U.P.T.U.2015)
87	$$\frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=r(t) \text{ where , } r(t)=\begin{cases} e^t & ,0<t<2 \\ 0 & ,t>2 \end{cases}$$ and x(0)=1 , x'(0)= - 2	[2010]
88	$\frac{d^2y}{dt^2}+9y=r(t)$ with initial conditions y (0) = 0 and y' (0) = 4, where $$r(t)=\begin{cases} 8\sin t & ,0<t<\pi \\ 0 & ,t>\pi \end{cases}$$	[2011]
89	Determine the response of damped mass-spring system under a square wave given by $y'' + 3y' + 2y = u(t-1) - u(t-2)$, $y(0) = 0, y'(0) = 0$ using the laplace transform.	[2013,2017]
90	Solve the Simultaneous equation by using Laplace transform : $$\frac{dx}{dt}+\frac{dy}{dt}+x+y=1, \frac{dy}{dt}=2x+y; x(0)=, y(0)=1.$$	[2011]